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A Hierarchical Model for the Reliability of an Anti-aircraft Missile System

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Abstract

We describe a hierarchical model for assessing the reliability of multi-component systems. Novel features of this model are the natural manner in which failure time data collected at either the component or subcomponent level is aggregated into the posterior distribution, and pooling of failure information between similar components. Prior information is allowed to enter the model in the form of actual point estimates of reliability at nodes, or in the form of prior groupings. Censored data at all levels of the system are incorporated in a natural way through the likelihood specification. The methodology is illustrated with an example from an anti-aircraft missile system.

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ABSTRACT

We describe a hierarchical model for assessing the reliability of multi-component systems. Novel features of this model are the natural manner in which failure time data collected at either the component or subcomponent level is aggregated into the posterior distribution, and pooling of failure information between similar components. Prior information is allowed to enter the model in the form of actual point estimates of reliability at nodes, or in the form of prior groupings. Censored data at all levels of the system are incorporated in a natural way through the likelihood specification. The methodology is illustrated with an example from an anti-aircraft missile system.

Keywords: aggregation error, Bayesian, expert opinion, failure time model, MCMC, multicomponent system, system reliability.

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1 Background

Estimating the reliability of complex systems such as missile systems and automotive systems is a challenging statistical problem. Perhaps the most difficult aspect of system reliability assessments is the integration of multiple sources of information, including component, subsystem and system data, as well as prior expert opinion. In addition, it is often necessary to infer how reliability changes over time. Such inferences are used to make predictions of system reliability for the purpose of setting warranties (e.g., in the case of automotive systems) and/or shelf-life (e.g., in the case of missile systems). While much attention has been paid to theoretical system reliability (Barlow and Proschan 1975) and empirical component reliability, there are few instances where these disparate approaches have been combined to model full system reliability when data have been collected at both the component and system level. In this paper, we propose a framework for achieving this synthesis by addressing two important analytical concerns: (1) the integration of available information at various levels to assess system reliability, and (2) estimating reliability growth or degradation. Methodology for achieving this integration has historically proven elusive; our resolution to this problem is based on the specification a Bayesian hierarchical model that accomodates both the inclusion of multiple information sources, and a convenient context for modeling the time evolution of a system's (or group of systems') reliability growth. Our model extends results presented in Johnson et al (2003), in which a hierarchical model for the (binary) success or failure of systems and their components was described.

1.1 Previous Work

To provide context, it is useful to begin with a review of related research in Bayesian system reliability. Most relevant to the model considered here are the papers by Martz, Waller and Fickas (1988) and Martz and Waller (1990), where complex systems, comprised of series and parallel subcomponents, were modeled using beta priors and binomial likelihoods at component, subsystem and system levels. Within this framework, an "induced" higher-level prior was obtained by propagating lower-level posteriors up through the system fault diagram, and combining these posteriors with "native" higher-level priors to obtain an induced prior at the next system level. The induced priors were then approximated by beta distributions using a methods-of-moments type procedure. The combination of native priors and posterior distributions obtained from lower-level system data, both of which were expressed as beta distributions, was accomplished by expressing the induced priors as a beta distributions with parameters representing a weighted average of the constituent beta densities. This process was propagated through subsequent system levels to obtain an approximation to the posterior distribution on the total system reliability. Johnson *et al* (2003) presents work on combination of multi-level binomial data. They employ a substitution principle in the same spirit as the modeling approach considered here.

Many common reliability models are not able to account for prior expert opinion and data when such information is simultaneously obtained at several levels within a system. Among those models that can accommodate such sources of information are those proposed by Springer and Thompson (1966, 1969), and Tang, Tang and Moskowitz (1994, 1997), who provide exact (and in complicated settings, approximate) system reliability distributions based on binomial data by propagating component posteriors through the system's reliability block diagram. Others have proposed methods for evaluating or bounding moments of the system reliability posterior distribution (Cole (1975), Mastran (1976), Dostal and Iannuzzelli (1977), Mastran and Singpurwalla (1978), Barlow (1985), Natvig and Eide (1987), Soman and Misra (1993)). Moment estimators have also been used in the beta approximations employed by Martz, Waller and Fickas (1988) and Martz and Waller (1990). In a somewhat different approach, Soman and Misra (1993) proposed distributional approximations based on maximum entropy priors.

Numerous models have, of course, also been proposed for modeling non-binomial data. Thompson and Chang (1975), Chang and Thompson (1976), Mastran (1976), Mastran and Singpurwalla (1978), Lampkin and Winterbottom (1983), and Winterbottom (1994) considered models for exponential lifetime data, while Hulting and Robinson (1990, 1994) examined Weibull models. We extend the methods proposed there to include a hierarchical specification on the nodes appearing in a reliability block diagram. Poisson count data, representing the number of units failing in a specified period, are discussed in Hulting and Robinson (1990), Sharma and Bhutani (1992), Hulting and Robinson (1994), Sharma and Bhutani (1994), and Martz and Baggerly (1997). Currit and Singpurwalla (1988) and Bergman and Ringi (1997a) considered dependence between components introduced through common operating environments. Bergman and Ringi (1997b) incorporated data from non-identical environments. However, the problem considered here—the combination of multi-level failure time data—has, to the best of our knowledge, not been considered elsewhere.

Many degradation models for system reliability restrict attention to settings in which only system-level data are available (e.g., Fries and Sen (1996), Nolander and Dietrich (1994), and Sohn (1996)). An exception to this trend is Robinson and Dietrich (1988), who modeled component-level data collected during system development using exponential lifetime assumptions and decreasing failure rates. In this work, we employ models that directly address aging and estimate reliability growth or decay through the use of Weibull failure times.

We present a self-consistent model for system reliability. In Section 2, we propose a model for system reliability estimation that allows the inclusion of component, subsystem, and full system test data. That model is illustrated with an application to anti-aircraft missile system data in Section 3. The extension of the model to account for censored data is described in Section 4. We conclude with a summary of results and suggestions for future work in Section 5.

2 Model

To illustrate the baseline model, consider Figure 1, which depicts a simplified version of a reliability block diagram for an anti-aircraft missile system. The features illustrated in this figure include the composition of a system by multiple subsystems. In this case,

there are three subsystems: the missile round, the battery coolant subsystem (BCU), and an unspecified electronics subsystem (further details concerning the reliability block diagram for this system have been omitted for proprietary reasons). We illustrate two levels of system structure in this schematic, although additional levels of granularity can be included without difficulty. In general, we assume that failure time data and prior expert opinion are available at different levels of the system, and that our primary goal in modeling such systems is the evaluation of the system reliability function, $R_1(t|\theta_1)$, defined as the probability that the system (in this case component 1) will function beyond time t, given the value of a parameter vector θ_1 . More generally, we let $R_i(t|\theta_i)$ denote the reliability of the *i*th node in the reliability diagram, and we assume that $R_i(t|\theta_i)$ is a continuous and differentiable function of both time t and the reliability parameter θ_i . To simplify terminology, we henceforth call terminal nodes in the reliability diagram "components", nodes in the middle of the reliability diagram "subsystems," and the node at the top the "system." (We note that this terminology is not entirely standard and requires some care when more complicated reliability block diagrams are considered; in such cases the diagram is often broken apart and segregated by subsystems, which, in our terminology then become systems.) Other quantities of interest are the hazard function, which is the *instantaneous* probability of failure at time t,

$$h_i(t;\theta_i) = \lim_{\Delta t \to 0} \frac{Pr(t < T \le t + \Delta t)}{\Delta t}$$
$$= \frac{f_i(t|\theta_i)}{R_i(t|\theta_i)},$$

where $f_i(t|\theta_i)$ is the failure time probability density function of node *i*.

Several sources of information relevant to estimating system reliability are incorporated into our model framework. The first is failure time data collected at individual components. The second is failure time data collected at the system or subsystem level; such tests are important because they provide both information on subsystem functioning as well as adjustments to component reliabilities that must be made to account for changes in reliability associated with aggregation of components to subsystems and the requirement for nodes to function simultaneously. A third source of information takes



Figure 1: Reliability Fault Tree for Missile Reliability

the form of expert opinion regarding the reliability of particular nodes. A fourth, less precise source of information is expert opinion regarding the similarity of reliabilities of groups of components or subsystems within the given system or across different systems. For example, in the missile system depicted above, an expert may assert that the reliability of the BCU is similar to the reliability of a BCU in a related missile system, or that reliabilities of the missile round and BCU are similar. However, the expert may not have knowledge regarding the specific probability that any component within a group of similar components functions. Finally, we incorporate the statistical notion that components in the reliability block diagram may also be grouped into sets of comparably reliable components without the guidance of actual expert opinion. In the baseline model, such information is modeled via an exchangeability assumption on the parameters of the failure time distribution.

Nodes in the reliability diagram are labeled C_i , and the set D_i contains the m_i times at which data for C_i is observed. The set A_i contains all component children of C_i . The number of components (i.e., terminal nodes) in the system is denoted by n_c . We let $\Theta = \{\theta_i\}, i = 1, ..., n_c$ denote the collection of parameters describing the lifetime distributions of system components.

2.1 Likelihood Specification

System reliability problems typically have two types of information contributing to the likelihood: component tests and system/subsystem tests. We seek a model which provides flexibility for incorporating both types of information in a way that preserves the probabalistic constructs defined by the reliability block diagram. As stated above, this is not a trivial task, and combining data and prior information at different levels within a reliability diagram has often proven problematic from both the perspectives of computational tractability and model consistency. Our solution to this conundrum is to simply re-express system and subsystem lifetime distributions in terms of component lifetimes using deterministic relations derived from an examination of the reliability block diagram.

Based on these considerations, we assume that test data collected at the component level contributes to the likelihood function in the usual way. That is, a failure at time tat component C_i contributes a factor of $f_i(t|\theta_i)$ to the likelihood function.

Data collected at the subsystem or system level must be incorporated into the likelihood function through an examination of the reliability block diagram of the system. For example, for a series-only system or subsystem (i.e., a nonredundant system), the cumulative distribution function for subsystem C_i at time t may be expressed (suppressing dependence on model parameters θ)

$$F_i(t) = 1 - R_i(t)$$

= $1 - \prod_{j \in A(i)} R_j(t).$

Note that the product in this expression ranges over only those *components* that have C_i as a parent—intervening subsystem reliabilities need not be counted twice. The sampling density at time t implied by this expression is

$$f_{i}(t) = \frac{dF_{i}(t)}{dt}$$

$$= -\frac{d}{dt} \prod_{j \in A_{i}} (1 - F_{j}(t))$$

$$= \sum_{j \in A_{i}} f_{j}(t) \prod_{\substack{k \neq j \\ k \in A_{i}}} (1 - F_{k}(t)). \quad (1)$$

For a parallel-only system or subsystem (i.e., a system comprised entirely of mutually redundant components), the cumulative distribution function at time t is

$$F_i(t) = 1 - R_i(t)$$

= $1 - \prod_{j \in A_i} (1 - R_j(t)).$

The sampling density at time t for such a parallel system is thus

$$f_{i}(t) = \frac{dF_{i}(t)}{dt}$$

$$= \frac{d}{dt} \prod_{j \in A_{i}} F_{i}(t)$$

$$= \sum_{j \in A_{i}} f_{j}(t) \prod_{\substack{k \neq j \\ k \in A_{i}}} F_{k}(t).$$
(2)

Appropriate combinations and modifications of these expressions can be used to construct sampling densities for systems or subsystems composed of an arbitrary number of components in various configurations of parallel and series subsets. Furthermore, components need not follow the same lifetime distributions. For example, we might assume that one component follows an exponential distribution, while modeling another according to a Weibull distribution. This feature of our framework allows for substantial flexibility in modeling complex systems for which components are acquired from different manufacturers under different specifications.

A simple example of this methodology is provided by the anti-aircraft system with the reliability block diagram illustrated in Figure 1. This is a non-redundant system, so its cumulative distribution function can be expressed

$$F_1(t|\Theta) = 1 - \prod_{i=2}^{4} [1 - F_i(t|\theta_i)]$$
$$= 1 - \prod_{i=2}^{4} R_i(t|\theta_i),$$
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where $R_i(t|\theta_i)$ is the reliability function for component C_i . Differentiating, we find that

the sampling distribution of the system failure times can be written

$$f_1(t|\Theta_S) = \sum_{i=2}^{4} f_i(t|\Theta_i) \prod_{j \neq i} (1 - F_j(t|\Theta_i)).$$
(3)

This model is used to model system and component level test data collected on this system in Section 3.

2.2 Prior Information

In many applications, expert opinion plays an important role in assessing system reliability, particularly in large complex systems for which data collected on individual subcomponents may be sparse. Expert opinion may be available from several experts, each of whom may provide information regarding the reliability of different subsets of components, and the quality of information obtained from each may vary. Efficiently incorporating expert knowledge into estimates of system reliability can therefore be a complicated task. Our solution to this problem is to elicit information from experts in the form of psuedo-observations. That is, we ask each expert to provide a value for the failure time for each component for which he has information. We then treat these assessments as if they were observations from the sampling density, except that we also incorporate a parameter that represents the precision of the information solicited from each expert. Other approaches are, of course, possible. We might, for instance, elicit prior judgments on reliability parameters. However, our approach has proven to be convenient from a practical standpoint because experts are often able to express opinions on how long they think a component will last. They are less willing to express opinions on mean lifetimes or other abstract model parameters. Also, incorporating expert opinion in the form of psuedo-observations substantially simplifies statistical modeling in a setting that is already quite complex.

With these considerations in mind, we assume that the prior information obtained from expert m concerning the lifetime distribution of component C_i contributes a factor of

$$f_i(t_{im}|\Theta^i)^{N_m},\tag{4}$$

to the joint posterior density. In this expression, N_m represents the precision or weight assigned to information collected from expert m. Loosely speaking, N_m may be regarded as the number of observations assigned to the expert's assessment of the *i*th component's lifetime distribution. The value of N_m is not assumed to be known a priori, but we instead assign a prior distribution to its value. A posteriori, plausible values of N_m are estimated from their prior distributions and the consistency of the expert's assessment with observed data and other experts. For concreteness, we assume that each expert's precision parameter N_m is drawn from a gamma density with known parameters α_m and β_m , parameterized here as

$$g(N_m; \alpha_m, \beta_m) = \frac{\beta_m^{\alpha_m}}{\Gamma(\alpha_m)} N_m^{\alpha_m - 1} \exp(-\beta_m N_m).$$

To incorporate expert at the subsystem or system level, we use constructions similar to those used in defining the likelihood function. For example, if expert m provides a value of t_{im} for the failure of subsystem C_i , whose functioning requires the functioning of all components in D_i (i.e., a non-redundant subset of the system components), then this information is assumed to contribute a factor of

$$f_i(t_{im}|\theta_i)^{N_m} = \left[\sum_{j \in A_i} f_j(t) \prod_{\substack{k \neq j \\ k \in C_i}} (1 - F_k(t))\right]^{N_m}$$

to the prior density on Θ . Similar expressions for parallel systems may be based on (2) for completely redundant systems.

Note that by eliciting prior information from the same expert at the component and system level, it is possible to partially assess the consistency of the expert's opinions with the assumed reliability block diagram, and to thus infer plausible values of N_m .

2.3 Hierarchical assumptions

Because expert opinion enters the model in the form of weighted data, we are left with the freedom to impose a prior distribution on the model parameters Θ in a way that reflects our prior beliefs on the exchangeability of system components. We accomplish this by assuming that the values of θ_i for specified groups of components are drawn from a common distribution. If the hyperparameters of this distribution are denoted by η , then in many settings it is feasible to also specify a second level model on the parameters η and to estimate there values from data. Such a structure is illustrated in Section 3.

2.4 Joint Posterior Distribution

For the moment, we assume that test data is completely observed; we discuss the case of censored data in Section 4.

Let $\mathbf{D} = \{D_i\}$ denote the test data available for constructing the likelihood function, and let E_m denote the set of nodes for which expert m provides prior assessments, and suppose that there are M experts from whom information has been solicited. Then under the assumptions described in previous sections, the joint posterior distribution on model parameters is proportional to

$$f(\Theta, \eta, \zeta | \mathbf{D}) \propto \prod_{i=1}^{M} \prod_{t \in D_i} [f_i(t_{im} | \theta_i)] \\ \times \prod_{m=1}^{M} \left[g(N_m | \alpha_m, \beta_m) \prod_{i \in E_m} f_i(t_{im} | \theta_i)^{N_m} \right] \\ \times \pi(\Theta | \eta) \pi(\eta) \prod_{m=1}^{M} \pi(\alpha_m, \beta_m | \eta)$$
(5)

where $\pi(\Theta|\eta)$ is the hierarchical prior specification of the parameters for the terminal node failure time distributions and $\pi(\eta)$ is the prior distribution on the η .

In (5), values of non-terminal node probabilities are assumed to be expressed in terms of the appropriate functions of terminal node probabilities, as defined from the system reliability block diagram.

2.5 Estimation strategies for the baseline model

The joint distribution of model parameters specified in (5) does not lend itself to analytical evaluation of the system or component reliabilities. However, a component-wise Metropolis-Hastings algorithm can be implemented in a relatively straightforward way. In

Component	Data
System	23.9, 18.0, 53.1, 27.6, 53.7, 34.5, 47.2, 25.7, 20.8, 7.1
Round	$5.3,\!65.9,\!15.5,\!39.4,\!47.2,\!28.2,\!91.7,\!33.6,\!13.4,\!13.9$
	117.7, 29.3, 35.5, 4.4, 150.4, 15.7, 47.0, 5.1, 23.5, 25.1
BCU	65.5, 51.9, 120.2, 32.0, 51.5, 70.5, 37.7, 9.7, 78.0, 24.9
	47.7, 46.6, 105.8, 70.5, 39.9, 29.8, 48.3, 25.4, 17.7, 27.6
C_4	28.8, 51.3, 41.2, 59.2, 19.9, 57.5, 64.4, 15.7, 75.0, 35.2
	57.5, 49.2, 18.2, 48.8, 57.5, 35.7, 29.4, 14.6, 46.2, 9.0

Table 1: Missile system test data. Observations are in units of tens of hours.

our version of such a scheme, we used a random-walk Metropolis-Hastings algorithm with Gaussian proposal densities for the terminal node probabilities, precision parameters, and hyperparameters of the hierarchical specification.

3 An Application to an Anti-aircraft Missile System

As a simple demonstration of the proposed methodology, consider the missile system described in Section 2. The reliability fault diagram for this system is depicted in Figure 1, which shows that this system consists of three non-redundant components. There are thus four reliability functions of interest, one for each of the components, and the system reliability function.

Test data available for estimating the reliability functions for this system are provided in Table 1. Twenty tests were conducted for each component, and ten system tests were performed. Failure times for each test are depicted in the table.

Two experts provided prior assessments for the system or component reliabilities.

Component	Expert	Lifetime
System	E1	32.0
System	E2	48.0
Round	E1	75.0
Round	E2	90.0
C_4	E3	70.0

Table 2: Expert Opinion for anti-aircraft missile example.

Expert 1 (E1) provided lifetime information about the full system and the missile round. E1 estimated that a system would survive 320 hours (or 32.0), and that a missile round would have a lifetime of 750 hours (or 75.0). Another expert (E2) provided information about the full system, the missile round and component 4 (C_4). This expert expected a full system to survive 480 hours (48.0). She also expected the missile round to survive 900 hours (90.0) and C_4 to survive 700 hours (70.0). No expert opinion is available for the BCU. A summary of the expert opinion data is shown in Table 2.

In this application, we use a Weibull distribution to model the component failure times. Our parameterization of the Weibull density for failure times for component C_i , i = 2, 3, 4, is

$$f_i(t|\psi,\lambda) = \frac{\psi_i}{\lambda_i} \left(t/\lambda_i\right)^{\psi_i - 1} \exp\left[-\left(t/\lambda_i\right)^{\psi_i}\right],\tag{6}$$

so that $\theta_i = (\psi_i, \lambda_i)$. Our prior specification for Θ (i.e., $\pi(\Theta|\eta)$) in this example is that the ψ_i and λ_i are conditionally independent given $(\delta_{\psi}, \zeta_{\psi})$ and $(\delta_{\lambda}, \zeta_{\lambda})$, respectively, and that all values of (ψ_i, λ_i) are drawn mutually independently from gamma distributions; that is,

$$\pi(\psi_i | \delta_{\psi}, \zeta_{\psi}) \propto \psi_i^{\delta_{\psi} - 1} \exp\left(-\zeta_{\psi} \psi_i\right),$$
$$\pi(\lambda_i | \delta_{\lambda}, \zeta_{\lambda}) \propto \lambda_i^{\delta_{\lambda} - 1} \exp\left(-\zeta_{\lambda} \psi_i\right).$$

To complete the hierarchical specification, we assume that $\delta_{\psi}, \zeta_{\psi}, \delta_{\lambda}, \zeta_{\lambda}$ have independent prior exponential distributions with mean 1.

We assigned a $g(\cdot | 5, 1)$ prior density to the expert opinion precision parameters N_1 and N_2 . A priori, this means that we value each expert's assessment to be worth approximately 5 observations. The posterior distribution on these precision parameters are examined below.

To sample from the posterior distribution on model parameters and reliabilities, we used a successive substitution Markov chain Monte Carlo (MCMC) procedure (Gelfand and Smith 1990), where each component of the joint posterior distribution was updated one-at-a-time. The posterior distributions that are presented below were based on 1,000,000 draws from the joint posterior distribution with a 100,000 burn-in period.

A plot of the marginal posterior densities on the reliabilities of components at different levels within the system is depicted in Figure 1. We note that the Weibull distribution has the following properties:

- if $\psi_i > 1$, the lifetime has an increasing failure rate,
- if ψ_i = 1, the lifetime has a constant failure rate (that is, the lifetime is exponentially distributed),
- if $\psi_i < 1$, the lifetime has a decreasing failure rate.

For these data, these properties imply that the posterior probability of an increasing failure rate is approximately 1.0 for all three components.

The prior distribution and posterior distributions for the expert precision parameters are depicted in Figure 4. These plots suggest that assessments from E1 were more consistent with observed data than were those from E2.

4 Extension to censored observations

The contribution of a right-censored observation to the likelihood function is the reliability function evaluated at the censored value (1 - F(t)) at the appropriate level in the reliability block diagram. The contributions of other forms of censoring are listed Table 3. Incorporating censored data into our model framework is thus straightforward and



Figure 2: Marginal posterior distributions of the parameters of the failure time distributions for each of the terminal nodes system represented in Figure 1. The left column represents the posterior distribution of ψ_i for each of the three components and the right column are the posterior distributions of λ_i .

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$$(a) (b)$$

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(c)
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(d)

Figure 3: Posterior distributions (as a function of time) for the reliability function of each of the components in the system represented in Figure 1. They are organized as (a), the posterior distribution of the full system C_1 , (b) is the posterior distribution for the missile round reliability, (c) is the posterior distribution for the BCU, and (d) is the posterior distribution for the unnamed component C_5 .

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Censoring Type	Likelihood Contribution
Uncensored	$f_i(t heta)$
Right Censored $(t > t_R)$	$1 - F_i(t_R \theta)$
Left Censored $(t < t_L)$	$F_i(t_L heta)$
Interval Censored $(t_L \le t \ge t_R)$	$F_i(t_R \theta) - F_i(t_L \theta)$

Table 3: Likelihood contributions of various types of censored data.

can be accomplished by simply substituting the appropriate expression for the censored observation from Table 3 for the sampling density of an observed failure in (5).

4.1 Diagnostics

Our model for system reliability is relatively complex and contains a number of assumptions that should be verified. Although a comprehensive examination of model diagnostics falls beyond the scope of this paper, we do stress the importance of such procedures. For present purposes, however, we restrict attention to the global model diagnostic proposed in Johnson (2004). This diagnostic can be considered as a Bayesian version of Pearson's chi-squared goodness-of-fit test.

The diagnostic requires that observations be conditionally independent given the value of the parameter vector Θ , which they are in our application. Let $\tilde{\Theta}$ denote a single value of the parameter vector drawn from the posterior distribution, and let u_j , $j = 1, \ldots, n$, be defined as

$$u_j = F(y_j | \Theta)$$

Then from results in Johnson (2004), it follows that the distribution of the chi-squared statistic obtained by assuming that the values of u_j are drawn from a uniform distribution on (0,1) has a chi-squared distribution on K-1 degrees of freedom, where K is the number of bins used to define the chi-squared statistic. No adjustment to the degrees of freedom need be made to account for the dimension of Θ .

To apply this procedure to our model for the missile data, we choose 5 equiprobable bins and calculated 10,000 chi-squared statistics based on 10,000 posterior draws of $\tilde{\Theta}$. Only 4.3% of these values exceeded the .95 quantile of a chi-squared distribution on 4 degrees of freedom, suggesting no lack of model fit based on this global diagnostic.

5 Conclusions

Our hierarchical model for system reliability offers several advantages over other existing models for system reliability. Among these are the ease of including diverse sources of information at different levels of the system in the model for overall system reliabilities, a coherent framework for incorporating multiple sources of prior expert opinion through the treatment of expert opinion as (imprecisely-observed) data, and the natural elimination of aggregation errors through the definition of non-terminal node probabilities using the assumed structure of the system reliability block diagram and terminal node failure time distributions.

In the setting where there are few, or perhaps no, system tests available, the borrowing of strength across nodes allows decision makers to use existing data in a more efficient manner. Hulting and Robinson (1994) discuss the reliance on elicited prior and their reservation that the conclusions about reliability of some subsystems relies entirely on the quality of the elicitation. Our hierarchical specification allows the incorporation of more vaguely specified prior information based on groupings of component nodes based on similarity of reliability, rather than more specific specifications of reliability values. This reliance on elicited priors is thereby shifted more to structural similarity of components and observed data, which we feel is an important innovation of our method. Johnson *et al* (2003) also discuss the benefits of such hierarchical specifications.

A very simple example of our hierarchical model for reliability was described in this paper. In future work we plan to extend this framework to include non-serial systems and extensions of the model to account for dependencies between component failures within a system or subsystem.

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(a) (b)

Figure 4: Posterior distributions for N_1 and N_2 for the anti-aircraft missile example. E1 whose posterior distribution for precision, N_1 is shown in (a) is more consistent than the apriori values, suggesting more precision or agreement with the system structure and observations, while E2 (shown in (b)) demonstrates less precision.

