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FORMULAS FOR THE EXACT PROBABILITY OF CORRECT SELECTION IN THE BINOMIAL LEVIN-ROBBINS SEQUENTIAL SELECTION PROCEDURE IN THE CASES b=2, c=3 AND b=2, c=4 FOR r=1

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1. Introduction.

The purpose of this report is to record some explicit expressions for the probability of correct selection in the Levin-Robbins procedure for selecting the best b out of c coins in the two cases in which such expressions are feasible. The reader is referred to Leu and Levin (2004) for terminology and notation.

2. Exact probability of correct selection of best two out of three coins without elimination of coins ($b = 2,\ c = 3$) in the case r = 1.

Remark 1. This problem is equivalent to selecting the worst coin. Also, by parity reversal, the results of this section apply to selecting the best coin without elimination.

Procedure A: Start with tallies (0,0,0), toss vector-at-a-time until for the first time $X_{(2)}^{(n)} - X_{(3)}^{(n)} = r = 1$, where $X_{(1)}^{(n)} \geq X_{(2)}^{(n)} \geq X_{(3)}^{(n)}$. Select coins corresponding to $X_{(1)}^{(n)}$ and $X_{(2)}^{(n)}$ (call these coins (1) and (2)).

Remark 2. This procedure can be represented as a Markov chain with an infinite state space, because $X_{(1)}^{(n)}$ can be arbitrarily large at stopping time. It is thus key to have the following result on returns to starting configurations.

Let the coins have probability of heads $p_i = 1 - q_i$ $(i = 1, 2, 3)$. Without loss of generality, we may assume $p_1 \geq p_2 \geq p_3$. Let $P = p_1p_2p_3$ and $Q = q_1q_2q_3$. Note that $P \neq 1 - Q$. Also, let $w_i = p_i/q_i$ $(i = 1, 2, 3)$.

Definition : Let the tally of heads after n tosses be $(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$. The *configuration* of the tally is $(X_1^{(n)} - X_{(3)}^{(n)}, X_2^{(n)} - X_{(3)}^{(n)}, X_3^{(n)} - X_{(3)}^{(n)})$.

Lemma 1. Suppose the tally configuration is $(0, 0, 0)$. Let E_0 be the event of an eventual return to $(0, 0, 0)$ before stopping. Then $P(E_0) = 3P(E)$, where

$$P(E) = \sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{m+1} \{\frac{PQ}{(1-P-Q)^2}\}^{(m+1)}. \tag{1}$$

Proof : Throughout we condition on vector outcomes that exclude all heads or all tails, so that the configuration changes at each step. The conditional probability of a step in the "outward" direction $(1, 0, 0)$, say, is $w_1Q/(1 - P - Q)$, while a step in the "inward" direction $(0, 1, 1)$ is $w_2w_3Q/(1 - P - Q)$. The lemma follows from counting paths on the linear lattices $\{(s, 0, 0) : s \geq 0\}$, $\{(0, s, 0) : s \geq 0\}$, and $\{(0, 0, s) : s \geq 0\}$. Consider a configuration at $(1, 0, 0)$. The number of paths that return to that configuration (not necessarily for the first time) in exactly $2m$ steps without stopping and without visiting (0,0,0) is $\frac{\binom{2m}{m}}{m+1}$ (see Lemma 2 below). Each such path has probability $\{w_1Q/(1 - P - Q)\}^m\{w_2w_3Q/(1 - P - Q)\}^m = \{PQ/(1 - P - Q)^2\}^m$. Thus the probability of an eventual return to $(1, 0, 0)$ before stopping is $\sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{m+1} \{\frac{PQ}{(1-P-Q)^2}\}^m$. If the random walk starts at $(0,0,0)$, takes a first step to (1,0,0), eventually returns to (1,0,0), and then takes a step back to $(0,0,0)$, the first and last steps introduce another factor of $PQ/(1-P-Q)^2$. The sum must begin at $m = 0$ to include an immediate return to $(0,0,0)$. Note that exactly the same expression results for sojourns in the initial direction $(0,1,0)$ or $(0,0,1)$, because in each case a step outward and a step inward have joint probability $\{w_2 Q/(1-P-Q)\}\{w_1 w_3 Q/(1-P-Q)\} = \{w_3 Q/(1-P-Q)\}\{w_1 w_2 Q/(1-P-Q)\}$ $= PQ/(1-P-Q)^2$. Thus $P(E_0) = 3 \sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{m+1} \{\frac{PQ}{(1-P-Q)^2}\}^{(m+1)}$. QED

Remark 3. The proof shows that if the starting configuration is $(0,0,0)$ and E_1 is the event of an initial step outward to $(1,0,0)$ followed by eventual return to $(0,0,0)$, and similarly for E_2 with initial step $(0,1,0)$ and E_3 with initial step $(0,0,1)$, then $P(E_1) = P(E_2) = P(E_3) = (1/3)P(E_0)$. We write the common value simply as $P(E) = \sum_{m=0}^{\infty} \frac{\binom{2m}{m}}{m+1} \{\frac{PQ}{(1-P-Q)^2}\}^{(m+1)}$ as in (1).

Remark 4. We have been tacitly assuming all $p_i > 0$. If any $p_i = 0$, then $P(E_1) = P(E_2) = P(E_3) = P(E) = P(E_0) = 0$ because inward steps along possible lattices have zero probability.

The number of paths returning to configuration $(1,0,0)$ is a special case of the following lemma.

Lemma 2. The number of paths starting at configuration $(s+1,0,0)$ for $s \geq 0$ and visiting configuration $(1,0,0)$ (not necessarily for the first time) in exactly $n \geq 0$ steps without visiting (0,0,0) is $\binom{n}{t}\frac{s+1}{t+1}$, where $t = (n+s)/2$.

Proof: (by the André reflection principle). Consider the two-dimensional lattice (n, s) from configuration $(s+1, 0, 0)$ after n steps (see Figure 1). There are $\binom{n}{t}$ lattice paths in total from $(0, s)$ to $(n, 0)$ at diagonally opposite corners of a rectangle. The number of these paths that cross or touch the line at level $s = -1$ and end at $(n, 0)$ equals the number of paths that start at $(0, s)$ and end at $(n, -2)$. There are $\binom{n}{t+1}$ of these. Thus the number of allowable paths that do not go below $s = 0$ equals $\binom{n}{t} - \binom{n}{t+1} = \binom{n}{t}(1 - \frac{n-t}{t+1}) = \binom{n}{t}(\frac{2t-n+1}{t+1}) = \binom{n}{t}(\frac{s+1}{t+1})$.

[Figure 1 here]

Theorem: For the three-coin procedure A to select the $b = 2$ coins with highest p_i, the probability of correct selection, $P[CS] = P[\{(1),(2)\} = \{1,2\}]$ is given by

$$P[CS] = \frac{\frac{w_1 w_2 Q}{1-P-Q}\{1 + \frac{1}{w_1+w_3} + \frac{1}{w_2+w_3}\} - \{\frac{w_1}{w_1+w_3} + \frac{w_2}{w_2+w_3}\}P(E)}{1 - 3P(E)}, \tag{2}$$

where $P(E)$ is given in (1).

Proof: It suffices to consider only paths that do not return to configuration $(0,0,0)$; formally, $P[CS] = P(E_0)P[CS] + \{1 - P(E_0)\}\ P[CS|$ no return to $(0,0,0)]$ by the stationarity of the procedure once it returns to the origin. Thus $P[CS] = P[CS|$ no return to $(0,0,0)]$. Now $P[CS|$ not $E_0] = \{P[CS|$ first step to $(1,0,0)$ and not $E_0]$ $P[(1,0,0)$ and not $E_0]$ + $P[CS|$ first step to $(0,1,0)$ and not $E_0]$ $P[(0,1,0)$ and not $E_0]$ + $P[$ first step is $(1,1,0)]\}/P[$ not $E_0]$. On the event $[(1,0,0)$ and not $E_0]$ the conditional probability of correct selection is $w_2/(w_2+w_3)$, corresponding to the event that coin 2 gets a head before coin 3, in which case the terminal configuration is $(s+1,1,0)$ for some $s \geq 0$. Furthermore, $P[(1,0,0)$ and not $E_0] = P[(1,0,0)$ and not $E_1] = P[(1,0,0)$ on the first step$] - P[E_1] = \frac{w_1 Q}{1-P-Q} - P[E]$. Similarly, $P[CS|(0,1,0)$ and not $E_0] = w_1/(w_1+w_3)$ and $P[(0,1,0)$ and not $E_0] = \frac{w_2 Q}{1-P-Q} - P(E)$. Also $P[(1,1,0)$ on first step $] = \frac{w_1 w_2 Q}{1-P-Q}$. Therefore

$$
\begin{aligned}
P[CS] &= \frac{\frac{w_2}{w_2+w_3}\{\frac{w_1 Q}{1-P-Q}-P(E)\}+\frac{w_1}{w_1+w_3}\{\frac{w_2 Q}{1-P-Q}-P(E)\}+\frac{w_1 w_2 Q}{1-P-Q}}{1-P(E_0)} \\
&= \frac{\frac{w_1 w_2 Q}{1-P-Q}\{1+\frac{1}{w_1+w_3}+\frac{1}{w_2+w_3}\}-\{\frac{w_1}{w_1+w_3}+\frac{w_2}{w_2+w_3}\}P(E)}{1-3P(E)}.
\end{aligned}
$$

Remark 5. We know by Remark 1 that $P[CS]$ is the same as that of the Levin-Robbins procedure without elimination for selecting the "best" coin (after parity reversal) and thus satisfies $P[CS] \geq \frac{w_3^{-1}}{w_1^{-1}+w_2^{-1}+w_3^{-1}} = \frac{w_1 w_2}{w_1 w_2+w_1 w_3+w_2 w_3}$, although this does not appear to follow trivially from (2).

Remark 6.

$$\frac{Q}{1-P-Q} = \frac{q_1q_2q_3}{p_1q_2q_3+q_1p_2q_3+q_1q_2p_3+p_1p_2q_3+p_1q_2p_3+q_1p_2p_3}$$
$$= \frac{1}{w_1+w_2+w_3+w_1w_2+w_1w_3+w_2w_3},$$

and

$$\frac{P}{1-P-Q} = \frac{(P/Q)Q}{1-P-Q}$$
$$= \frac{w_1w_2w_3}{w_1+w_2+w_3+w_1w_2+w_1w_3+w_2w_3}$$
$$= \frac{1}{w_1^{-1}+w_2^{-1}+w_3^{-1}+w_1^{-1}w_2^{-1}+w_1^{-1}w_3^{-1}+w_2^{-1}w_3^{-1}}.$$

If $p_1 = p_2 = p_3 = p$, say, and $w = p/q$, then $\frac{Q}{1-P-Q} = \frac{1}{3(w+w^2)}$, and

$$P[CS] = \frac{\frac{w^2}{3(w+w^2)}\{1+\frac{1}{2w}+\frac{1}{2w}\} - P(E)}{1-3P(E)} = \frac{1/3-P(E)}{1-3P(E)} = 1/3.$$

If $p_3 = 0$, since $P(E) = 0$, then

$$P[CS] = \frac{w_1w_2}{w_1+w_2+w_1w_2}\{1+\frac{1}{w_1}+\frac{1}{w_2}\} = \frac{w_1+w_2+w_1w_2}{w_1+w_2+w_1w_2} = 1.$$

If $p_1 = 0$ or $p_2 = 0$, $P[CS] = \frac{0-0}{1-0} = 0$.

If $p_1 = 1$, we must interpret $w_1Q = p_1q_2q_3 = q_2q_3$, $P = p_2p_3$, $Q = 0$, $w_1w_2Q = p_2q_3$ and $\frac{w_1}{w_1+w_3} = 1$. Then

$$\begin{aligned} P[CS] &= \tfrac{p_2 q_3}{1-p_2 p_3}\{1+\tfrac{1}{w_2+w_3}\} = \tfrac{p_2 q_3\{1+\frac{1}{w_2+w_3}\}}{q_2 q_3+p_2 q_3+q_2 p_3} \\ &= \tfrac{w_2}{1+w_2+w_3}\{\tfrac{1+w_2+w_3}{w_2+w_3}\} = \tfrac{w_2}{w_2+w_3}. \end{aligned}$$

Similarly, for $p_2 = 1$, interchange the role of coins 1 and 2, $P[CS] = \frac{w_1}{w_1+w_3}$. If $p_3 = 1$ then $Q = 0$, which implies $P[CS] = 0$.

Remark 7. Suppose the selection procedure starts at configuration $(s+1, 0, 0)$ for $s \geq 0$, as could occur with a four-coin procedure with elimination ($c = 4, b = 2, r = 1$). Let $E_{1,s} =$ [return to configuration $(0, 0, 0)$ before stopping]. From Lemma 2, there are $\binom{n}{t}\frac{s+1}{t+1}$ paths that return to configuration $(1, 0, 0)$ (not necessarily for the first time) in exactly n steps, where $t = (n+s)/2$, and each such path has probability $(\frac{w_1 Q}{1-P-Q})^{\frac{n-s}{2}} (\frac{w_2 w_3 Q}{1-P-Q})^{\frac{n+s}{2}} = (\frac{w_2 w_3}{w_1})^{\frac{s}{2}} \{\frac{PQ}{(1-P-Q)^2}\}^{\frac{n}{2}}$, assuming n has the same parity as s. Thus if $s = 2u$ and $n = 2m$, paths have probability $(\frac{w_2 w_3}{w_1})^u \{\frac{PQ}{(1-P-Q)^2}\}^m$, and if $s = 2u+1$ and $n = 2m+1$ ($u \geq 0, m \geq 0$), paths have probability

$$\begin{aligned} & (\tfrac{w_2 w_3}{w_1})^{u+\frac{1}{2}}\{\tfrac{PQ}{(1-P-Q)^2}\}^{m+\frac{1}{2}} \\ = \; & (\tfrac{w_2 w_3}{w_1})^{u}\{\tfrac{PQ}{(1-P-Q)^2}\}^{m}\{\tfrac{w_2 w_3}{w_1}\tfrac{PQ}{(1-P-Q)^2}\}^{\frac{1}{2}} \\ = \; & (\tfrac{w_2 w_3}{w_1})^{u}\{\tfrac{PQ}{(1-P-Q)^2}\}^{m}\tfrac{w_2 w_3 Q}{1-P-Q} \end{aligned}$$

Therefore, including the final step to configuration $(0, 0, 0)$ we have for $s = 2u$,

$u \geq 0$,

$$P[E_{1,2u}] = \sum_{m=0}^{\infty} \binom{2m}{m+u} \frac{2u+1}{m+u+1} \{\frac{PQ}{(1-P-Q)^2}\}^m \frac{w_2 w_3}{w_1})^u \frac{w_2 w_3 Q}{1-P-Q} \quad (3)$$

and for $s = 2u+1$, $u \geq 0$,

$$P[E_{1,2u+1}] = \sum_{m=0}^{\infty} \binom{2m+1}{m+u+1} \frac{2u+2}{m+u+2} \{\frac{PQ}{(1-P-Q)^2}\}^m (\frac{w_2 w_3}{w_1})^u (\frac{w_2 w_3 Q}{1-P-Q})^2 \quad (4)$$

Similarly, if $E_{2,s}$ = [return to configuration $(0,0,0)$ before stopping] given starting configuration $(0, s+1, 0)$, and $E_{3,s}$ likewise for $(0,0,s+1)$, then $P[E_{s,2}]$ follows (3) and (4) with the final factor replaced by $(\frac{w_1 w_3}{w_2})^u (\frac{w_1 w_3 Q}{1-P-Q})^{1 \text{ or } 2}$, and $P[E_{3,s}]$ replaces the final factor by $(\frac{w_1 w_2}{w_3})^u (\frac{w_1 w_2 Q}{1-P-Q})^{1 \text{ or } 2}$. Note in (3) and (4) the binomial coefficients can be written as $\binom{2m}{m-u}$ and $\binom{2m+1}{m-u}$, respectively. Therefore the first non-zero term in either sum is $m = u$.

Remark 8. Following logic similar to the Theorem, the probability of correct selection given initial configuration $(s+1,0,0)$, which we write as $P[CS|(s+1,0,0)]$, is

$$P[CS|(s+1,0,0)] = P(E_{1,s})P[CS|(0,0,0)] + \{1 - P(E_{1,s})\}\frac{w_2}{w_2 + w_3}$$

because, as before, on any sample path not in $E_{1,s}$ the probability of correct selection is simply $\frac{w_2}{w_2+w_3}$. Similarly, $P[CS|(0, s+1, 0)] = P(E_{2,s})P[CS|(0,0,0)] + (1 - P(E_{2,s}))\frac{w_1}{w_1+w_3}$ and $P[CS|(0,0,s+1)] = P(E_{3,s})P[CS|(0,0,0)]$, because paths starting from $(0,0,s+1)$ not in $E_{3,s}$ can not terminate in a correct selection.

3. Exact probability of correct selection of the best two out of four coins with elimination (c = 4, b = 2) in the case r = 1.

We now derive the $P[CS]$ for the Levin-Robbins procedure with elimination of inferior coins in the simplest case, $r = 1$. Without loss of generality, we assume the probabilities of heads are $p_1 \geq p_2 \geq p_3 \geq p_4$.

Procedure B : Sample coins $1, \cdots, 4$ vector-at-a-time until the time of first elimination, $N = inf\{n \geq 1 : X_{(2)}^{(n)} - X_{(4)}^{(n)} = r\}$, where $X_{(1)}^{(n)} \geq X_{(2)}^{(n)} \geq X_{(3)}^{(n)} \geq X_{(4)}^{(n)}$. At time N, any and all coins j with $X_j^{(N)} = X_{(4)}^{(N)}$ are eliminated from further consideration. If more than b coins remain, the procedure continues from the current tallies. Stopping occurs at time $N^* = inf\{n \geq 1 : X_{(2)}^{(n)} - X_{(3)}^{(n)} = 1\}$.

Stopping occurs at time $N^* = n$ if the configuration $\mathbf{a}^{(n)} = (X_1^{(n)} - X_{(4)}^{(n)}, X_2^{(n)} - X_{(4)}^{(n)}, X_3^{(n)} - X_{(4)}^{(n)}, X_4^{(n)} - X_{(4)}^{(n)})$ is of the form $(1, 1, 0, 0)$ or $(s+2, 1, 0, 0)$ or $(1, s+2, 0, 0)$ for $s \geq 0$ with a correct selection, or for other permutations of these configurations with an incorrect selection. The procedure continues for other configurations of the form $(1, 1, 1, 0)$ or $(s + 2, 1, 1, 0)$ for $s \geq 0$, or permutations thereof. For ease of notation, we write configuration events at time of first elimination as $[(1, 1, 0, 0)]$, for example, and the set of coins in play after N as C. The probability of correct selection $P[CS] = P[\{X_1^{(N^*)}, X_2^{(N^*)}\} = \{X_{(1)}^{(N^*)}, X_{(2)}^{(N^*)}\}]$ is given by expression (5):

$$
\begin{aligned}
P[CS] &= P[(1,1,0,0)] + P[(1,1,1,0)]P[CS|C=\{1,2,3\},(0,0,0)] \\
&+P[(1,1,0,1)]P[CS|C=\{1,2,4\},(0,0,0)] \\
&+\textstyle\sum_{s=0}^{\infty} P[(s+2,1,0,0)] + \sum_{s=0}^{\infty} P[(1,s+2,0,0)] \\
&+\textstyle\sum_{s=0}^{\infty} P[(s+2,1,1,0)]P[CS|C=\{1,2,3\},(s+1,0,0)] \\
&+\textstyle\sum_{s=0}^{\infty} P[(1,s+2,1,0)]P[CS|C=\{1,2,3\},(0,s+1,0)] \\
&+\textstyle\sum_{s=0}^{\infty} P[(1,1,s+2,0)]P[CS|C=\{1,2,3\},(0,0,s+1)] \\
&+\textstyle\sum_{s=0}^{\infty} P[(s+2,1,0,1)]P[CS|C=\{1,2,4\},(s+1,0,0)] \\
&+\textstyle\sum_{s=0}^{\infty} P[(1,s+2,0,1)]P[CS|C=\{1,2,4\},(0,s+1,0)] \\
&+\textstyle\sum_{s=0}^{\infty} P[(1,1,0,s+2)]P[CS|C=\{1,2,4\},(0,0,s+1)]. \qquad (5)
\end{aligned}
$$

We have already derived expressions for the $P[CS]$ in the reduced procedure after time of first elimination. It remains to derive the probability of the various configurations at time of first elimination.

As in Remark 3, starting from $(0,0,0,0)$, we define events $E_1^* =$ [first step to $(1,0,0,0)$ and return to $(0,0,0,0)$ before first elimination], $\cdots$, $E_4^* =$ [first step to $(0,0,0,1)$ and return to $(0,0,0,0)$ before first elimination]. These events all have probability equal to $P(E^*)$, say, with

$$P(E^*) = \sum_{m=0}^{\infty} \binom{2m}{m} \frac{1}{m+1} \{ \frac{P^* Q^*}{(1 - P^* - Q^*)^2} \}^{m+1}, \tag{6}$$

where $P^* = p_1p_2p_3p_4$ and $Q^* = q_1q_2q_3q_4$.

As in the proof of the preceding theorem, it suffices to consider sojourns that do not return to zero to derive the probability of configurations at time of first elimination, because the unconditional probability equals the conditional probability given no return to $(0,0,0,0)$. Thus, for example, $P[(1,1,0,0)] = P[(1,1,0,0)|$ no return to $(0,0,0,0)] = \{P[(1,1,0,0)$ on first step $]+P[(1,0,0,0)$ on first step followed by any path that returns to (1,0,0,0) without returning to (0,0,0,0) followed by outcome (0,1,0,0) $] + P[(0,1,0,0)$ on first step followed by any path that returns to (0,1,0,0) without returning to (0,0,0,0) followed by outcome (1,0,0,0) $]\}/(1-4P(E^*))$

$$\begin{aligned} &= \{\tfrac{w_1w_2Q^*}{1-P^*-Q^*} + P(E^*)\tfrac{w_2}{w_2w_3w_4} + P(E^*)\tfrac{w_1}{w_1w_3w_4}\}/(1-4P(E^*)) \\ &= \{\tfrac{w_1w_2Q^*}{1-P^*-Q^*} + \tfrac{2P(E^*)}{w_3w_4}\}/(1-4P(E^*)). \end{aligned} \tag{7}$$

The second and third terms follow because the event E_1^* (respectively, E_2^*) is isomorphic to paths that visit $(1,0,0,0)$ on the first step (respectively, $(0,1,0,0)$) and on the last step move to $(0,1,0,0)$ (respectively, $(1,0,0,0)$) instead of inward to $(0,0,0,0)$ in direction $(0,1,1,1)$ (respectively, $(1,0,1,1)$).

For other permutations of $(1,1,0,0)$, we interchange subscripts, or use the fundamental transposition lemma for configurations $\mathbf{a} = (a_1, a_2, a_3, a_4)$ at time of first elimination: if $\mathbf{a_{ij}}$ is $\mathbf{a}$ with a_i and a_j interchanged, then $P[\mathbf{a_{ij}}] = P[\mathbf{a}]w_{ij}^{-(a_i-a_j)}$, where $w_{ij} = w_i/w_j$ for any i, j. Thus,

$$P[1,0,1,0] = P[1,1,0,0] \cdot w_{23}^{-1},$$

$$P[1,0,0,1] = P[1,1,0,0] \cdot w_{24}^{-1},$$

$$P[0,1,1,0] = P[1,1,0,0] \cdot w_{13}^{-1},$$

$$P[0,1,0,1] = P[1,1,0,0] \cdot w_{14}^{-1},$$

$$P[0,0,1,1] = P[1,1,0,0] \cdot w_{13}^{-1}w_{24}^{-1} = P[1,1,0,0] \cdot w_{14}^{-1}w_{23}^{-1}$$

.

We argue similarly for $P[(1,1,1,0)]$: $P[(1,1,1,0)] = P[(1,1,1,0)|$ no return to $(0,0,0,0)] = \{P[(1,1,1,0)$ on first step $] + P[(1,0,0,0)$ on first step followed by any path returning to (1,0,0,0) but not (0,0,0,0), followed by outcome (0,1,1,0) $] + P[(0,1,0,0)$ on first step followed by any path returning to (0,1,0,0) but not

(0,0,0,0), followed by outcome (1,0,1,0)] + $P[(0,0,1,0)$ on first step followed by any path returning to (0,0,1,0) but not (0,0,0,0), followed by outcome (1,1,0,0)]}$/(1-4P(E^*))$

$$= \{\tfrac{w_1w_2w_3Q^*}{1-P^*-Q^*} + P(E^*)(\tfrac{w_2w_3}{w_2w_3w_4} + \tfrac{w_1w_3}{w_1w_3w_4} + \tfrac{w_1w_2}{w_1w_2w_4}\}/(1-4P(E^*))$$

$$= \{\tfrac{w_1w_2w_3Q^*}{1-P^*-Q^*} + \tfrac{3P(E^*)}{w_4}\}/(1-4P(E^*)). \qquad (8)$$

Similarly,

$$P[(1,1,0,1)] = \{\frac{w_1w_2w_4Q^*}{1-P^*-Q^*} + \frac{3P(E^*)}{w_3}\}/(1-4P(E^*)),$$

$$P[(1,0,1,1)] = \{\frac{w_1w_3w_4Q^*}{1-P^*-Q^*} + \frac{3P(E^*)}{w_2}\}/(1-4P(E^*)), \text{ and}$$

$$P[(0,1,1,1)] = \{\frac{w_2w_3w_4Q^*}{1-P^*-Q^*} + \frac{3P(E^*)}{w_1}\}/(1-4P(E^*)).$$

Next consider $P[(s+2,1,0,0)]$ for $s \geq 0$. Now we enumerate paths of two forms: (a) $(1,0,0,0)$ on first step followed by paths taking $s+m$ steps outward toward $(1,0,0,0)$ and m steps inward toward $(0,1,1,1)$, in any order, never returning to $(0,0,0,0)$, followed by $(1,1,0,0)$; and (b) $(1,0,0,0)$ on first step followed by paths taking $s+m+1$ steps toward $(1,0,0,0)$ and m steps toward $(0,1,1,1)$, followed by

$(0,1,0,0)$. Paths of type (a) have probability $(\frac{w_1Q^*}{1-P^*-Q^*})^{s+m+1}(\frac{w_2w_3w_4Q^*}{1-P^*-Q^*})^m \frac{w_1w_2Q^*}{1-P^*-Q^*}$, and paths of type (b) have probability $(\frac{w_1Q^*}{1-P^*-Q^*})^{s+m+2}(\frac{w_2w_3w_4Q^*}{1-P^*-Q^*})^m \frac{w_2Q^*}{1-P^*-Q^*}$.

Lemma 3. The number of paths starting with configuration $(1,0,0,0)$ and ending at configuration $(s+1,0,0,0)$, not necessarily for the first time and never visiting $(0,0,0,0)$, in exactly $s+2m$ steps is $\binom{s+2m}{m}\frac{s+1}{s+m+1}$. The number of paths starting with configuration $(1,0,0,0)$ and ending at configuration $(s+2,0,0,0)$, not necessarily for the first time and never visiting $(0,0,0,0)$, in exactly $s+2m+1$ steps is $\binom{s+2m+1}{m}\frac{s+2}{s+m+2}$.

Proof : In the two dimensional lattice (n,s) corresponding to configuration $(s+1,0,0,0)$ after n steps, by the André Reflection Principle, there are as many paths that start at $(0,0)$, cross or touch level -1, and end at $(s+2m,s)$ as there are paths that end as far below level -1 as s is above level -1, i.e., end at $(s+2m,-(s+2))$. Unrestricted paths with $m+s$ outward steps and m inward steps number $\binom{s+2m}{m}$ while unrestricted paths with $m-1$ outward steps and $s+m+1$ inward steps number $\binom{s+2m}{s+m+1} = \binom{s+2m}{m-1}$. Thus there are $\binom{s+2m}{m} - \binom{s+2m}{m-1} = \binom{s+2m}{m}(1-\frac{m}{s+m+1})$ $= \binom{s+2m}{m}(\frac{s+1}{s+m+1})$ paths that do not cross level -1, i.e., do not visit configuration $(0,0,0,0)$. For paths with $m+s+1$ outward steps and m inward steps, replace s by $s+1$. QED

Therefore we find $P[(s+2,1,0,0)] = P[(s+2,1,0,0)|$ no return to $(0,0,0,0)]$

$$\begin{aligned}
= & \quad \{\textstyle\sum_{m=0}^{\infty}\binom{s+2m}{m}\frac{s+1}{s+m+1}(\frac{P^*Q^*}{(1-P^*-Q^*)^2})^m(\frac{w_1Q^*}{1-P^*-Q^*})^{s+1}(\frac{w_1w_2Q^*}{1-P^*-Q^*}) \\
+ & \quad \textstyle\sum_{m=0}^{\infty}\binom{s+2m+1}{m}\frac{s+2}{s+m+2}(\frac{P^*Q^*}{(1-P^*-Q^*)^2})^m(\frac{w_1Q^*}{1-P^*-Q^*})^{s+2}(\frac{w_2Q^*}{1-P^*-Q^*})\} \\
\times & \quad (1-4P(E^*))^{-1} \\
= & \quad \{\textstyle\sum_{m=0}^{\infty}\binom{s+2m}{m}\frac{s+1}{s+m+1}(\frac{P^*Q^*}{(1-P^*-Q^*)^2})^m(\frac{w_1Q^*}{1-P^*-Q^*})^{s+2}\cdot w_2 \\
+ & \quad \textstyle\sum_{m=0}^{\infty}\binom{s+2m+1}{m}\frac{s+2}{s+m+2}(\frac{P^*Q^*}{(1-P^*-Q^*)^2})^m(\frac{w_1Q^*}{1-P^*-Q^*})^{s+3}\cdot\frac{w_2}{w_1}\} \\
\times & \quad (1-4P(E^*))^{-1} \qquad (9)
\end{aligned}$$

Similarly, for $P[(1,s+2,0,0)]$, interchange w_1 and w_2 in (9), or use the transposition lemma, $P[(1,s+2,0,0)] = P[(s+2,1,0,0)]w_{12}^{-(s+1)}$.

Next consider $P[(s+2,1,1,0)]$. Here we count paths of two types again (a) $(1,0,0,0)$ on first step followed by $s+m$ outward and m inward steps in any order, never returning to $(0,0,0,0)$, followed by $(1,1,1,0)$; and (b) $(1,0,0,0)$ on first step followed by $s+m+1$ outward and m inward steps in any order, never returning to $(0,0,0,0)$, followed by $(0,1,1,0)$. This is the same as for $(s+2,1,0,0)$ except for the final step. Thus $P[(s+2,1,1,0)]$ equals

$$\{\textstyle\sum_{m=0}^{\infty}\binom{s+2m}{m}\frac{s+1}{s+m+1}(\frac{P^*Q^*}{(1-P^*-Q^*)^2})^m(\frac{w_1Q^*}{1-P^*-Q^*})^{s+2}\cdot w_2w_3$$

$$+ \quad \sum_{m=0}^{\infty} \binom{s+2m+1}{m} \frac{s+2}{s+m+2} \left(\frac{P^*Q^*}{(1-P^*-Q^*)^2}\right)^m \left(\frac{w_1 Q^*}{1-P^*-Q^*}\right)^{s+3} \cdot \frac{w_2 w_3}{w_1} \}$$

$$\times \quad (1-4P(E^*))^{-1} \tag{10}$$

Similarly, for $P[(1, s+2, 1, 0)]$, interchange w_1 and w_2 in (10); for $P[(1, 1, s+2, 0)]$ interchange w_1 and w_3 in (10); for $P[(s+2, 1, 0, 1)]$ replace w_3 in (10) by w_4; for $P[(1, s+2, 0, 1)]$ replace w_3 by w_4 in (10) and interchange w_1 and w_2; for $P[(1, 1, 0, s+2)]$ replace w_3 by w_4 in (10) and interchange w_1 and w_4. Alternatively, use the transposition lemma, e.g., $P[(1, s+2, 1, 0)] = P[(s+2, 1, 1, 0)]w_{12}^{-(s+1)}$, and $P[(1, 1, 0, s+2)] = P[(s+2, 1, 1, 0)]w_{14}^{-(s+2)}w_{13}^{+1}$ etc.

Reference

Leu, C.S. and Levin, B. (2004). A generalization of the Levin-Robbins procedure for binomial subset selection and recruitment problems. (manuscript submitted)